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A VERY ABSOLUTE Π_2^1 REAL SINGLETON

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0. Introduction

By Shoenfield absoluteness theorem, the Π_2^1 real singleton are the simplest, non constructible, projective reals. Very few Π_2^1 non constructible real singletons exist in the literature.

One of them is the real $0^{\#}$ introduced by Solovay in [9]. It is 'provably Π_2^1 singleton', that means that there is a Π_2^1 formula φ such that for any model M of ZF that contains $0^{\#}$ as an element, $0^{\#}$ is — in M — the only solution of φ. The existence of $0^{\#}$ needs a large cardinal property as Ramsey or measurable and $0^{\#}$ is not generic over L by mean of a constructible set of conditions.

Other ones have been constructed by forcing over L (see Jensen and Solovay [8] and Jensen [5]). They are not 'provably Π_2^1 singleton'; they only are Π_2^1 singleton 'at home', that means there is a Π_2^1 formula φ such that the real r is the only solution of φ in $L(r)$, but (except in trivial cases — for instance if M and $L(r)$ have the same reals) it is not true that if M is another model containing r as an element, r remains — in M — the only solution of φ. Of course, by Shoenfield theorem, r remains a solution of φ but not necessarily the only one.

Later Jensen and Johnsbraten constructed, also by forcing, a new and more absolute Π_2^1 singleton r: r is the only solution — in M — of some Π_2^1 formula φ as far as M satisfies: $\omega_1 = \omega_1^L$.

In the sense of absoluteness, this real is the 'best' Π_2^1 singleton known, except $0^{\#}$. Other Π_2^1 singleton with various properties have been constructed (see [2, 3, 4]), but — in view of absoluteness — since they are constructed by use of the Jensen and Johnsbraten method, they are not different from that one.

A natural question is: find a 'provably' Π_2^1 non constructible singleton r such that $0^{\#} \notin L(r)$? It seems to be a difficult problem. This paper is devoted to the proof of a theorem which is — perhaps — a step to a solution of this problem.

Theorem 1. *Let M be a transitive model of* $\mathrm{ZF} + V = L$. *There is an M definable* *class P of forcing conditions and a Π^1_2 formula φ such that, if N is a P generic extension of M, then:*

(1) *N and M have the same cardinals and the same cofinality function.*

(2) *There is a real r in N such that N satisfies:*

$$\mathrm{ZF} + \neg 0^{\#} + V = L(r) + r \notin L + \varphi(r) + \exists !\, x\, \varphi(x).$$

(3) *If $\bar{N}$ is a generic extension of N (by a set of conditions), then $\bar{N}$ satisfies: $\exists !\, x\, \varphi(x)$.*

Note. (1) The real given by the theorem is not provably Π^1_2 singleton. I shall discuss and make some comments at the end of this paper.

(2) If $0^{\#}$ exists, I may assume that this real r is such that $r \in L(0^{\#})$; but r is not Π^1_2 singleton in $L(0^{\#})$.

(3) The conclusion of (3) in the theorem remains true for some other extensions of N: for instance if $\bar{N}$ is a generic extension of N by use of the class P of conditions where P is:

(i) the forcing to add a Cohen subset of $[\alpha, \alpha^{+}[$ for all the cardinals α,

(ii) the forcing to deny the generalized continuum hypothesis.

On the other way there are classes Q of conditions such that r does not remain Π^1_2 singleton in a Q generic extension of N.

The proof of this theorem will use the method developed by Jensen in his 'Coding the universe by a real' [6]. A basic and important modification has to be done, so the entire proof — and not only one or two points — has to be reexamined. It will be of course too long and fastidious — and useless — to rewrite the whole proof in this new context. So, in a first step I shall prove a theorem which is the basic idea of the theorem and the 'building block' of it. Then, while proving the theorem, I shall invite the reader to work with a copy — and/or a good knowledge — of Jensen's theorem by himself.

1. The building block

This section is devoted to the proof of the following:

Theorem 2. *Let M be a transitive model of $\mathrm{ZF} + V = L$. There is an M definable set P of conditions and a Π^1_2 formula φ such that if N is a P generic extension of M, then:*

(1) *N and M have the same cardinals and the same cofinality function.*

(2) *There is a real a in N such that N satisfies:*

$$V = L(a) + \varphi(a) + \exists !\, x\, \varphi(x).$$

(3) *If $\bar{N}$ is a generic extension of N by use of a set of conditions with the strong*

ω_2-chain condition (i.e.: $\forall\, x \subset C\ (\bar{\bar{x}} = \omega_2 \Rightarrow \exists\, y \subset x\ (\bar{\bar{y}} = \omega_2$ and $\forall\, c, c' \in y$, c and c' are compatible))), then $\bar{N}$ satisfies: $\exists!\, x\ \varphi(x)$.

This real comes from something as a three steps forcing extension. I work in a model $M_0 = M$ of $V = L$.

1.1. The trees

I define a sequence $(T(n))_{n\in\omega}$ of Suslin trees of height ω_2 in the following way (it is an easy generalisation of Jensen's construction of a Suslin tree of height ω_1). If T is a tree I shall denote T^α the αth level of T and $T \restriction \alpha = \bigcup_{\beta<\alpha} T^\beta$. The $T(n)$ will be such that: $x \in T^\alpha(n) \Rightarrow x : \alpha \to 2$. The levels of the $T(n)$ are defined by induction as follows:

$$x \in T^0(n) \quad \text{iff} \quad x = \emptyset,$$

$$x \in T^{\alpha+1}(n) \quad \text{iff} \quad x = y \cup \{(\alpha, i) \mid i = 0 \text{ or } 1\} \quad \text{for some } y \in T^\alpha(n_2)$$

for limit α,

$$\mathrm{cf}(\alpha) = \omega : x \in T^\alpha(n) \quad \text{iff} \quad x : \alpha \to 2 \ \&\ \forall\, \lambda < \alpha\ x \restriction \lambda \in T^\lambda(n)$$

$$(\text{since } \overline{\overline{\omega_2^\omega}} = \omega_1,\ \bar{\bar{T}}^\lambda(n) = \omega_1)$$

for limit λ,

$$\mathrm{cf}(\lambda) = \omega_1.$$

The elements of $T^\lambda(n)$ are branches in $T(n) \restriction \lambda$ choosen (in a classical way) by the following forcing:

Let η_λ be the least η such that $(T(n) \restriction \lambda)_{n\in\omega}$ is in L_η, and $L_\eta \vDash \mathrm{ZF}^- + \bar{\bar{\lambda}} = \omega_1$. Let

$$F_\lambda = \{(\alpha, f) \in L_{\eta_\lambda} \mid \alpha < \omega_1 \ \&\ f = (f_n)_n\, f_n : \alpha \to T(n) \restriction \lambda \quad \text{s.t.}$$
$$\forall\, n, m \in \omega\ \forall\, \beta, \bar{\beta} < \alpha \mid |f_n(\beta)| = |f_m(\bar{\beta})|\},$$

where $|x|$ is the level of x. Set $(\alpha, f) \leqslant (\bar{\alpha}, \bar{f})$ iff $\alpha \geqslant \bar{\alpha}$ and

$$\forall\, n \in \omega\ \forall\, \beta < \bar{\alpha}\ f_n(\beta) \geqslant \bar{f}_n(\beta)$$

(in the ordering of $T(n) \restriction \lambda$).

Because of the definition of $T(n)$ in the case of cofinality ω, it is clear that F_λ is an ω-closed set of conditions and so there is — in $\mathbb{L}$ — an F_λ generic over L_{η_λ}. The L-least one gives ω_2 branches in $T(n) \restriction \lambda$. They are the elements of $T^\lambda(n)$. This achieves the definition of the $T(n)$.

Lemma. *Let a be a finite subset of ω, and $T_a = \prod_{n\notin a} T(n)$. That is: a condition is a sequence $(x_n)_{n\in\omega-a}$ such that $x_n \in T(n)$; and $(x_n)_n \leqslant (y_n)_n$ iff $\forall\, n \in \omega - a\ x_n \geqslant y_n$. Then T_a is ω-closed and satisfies the ω_2-chain condition. Moreover let G be T_a generic over L and $i \in a$, then $L(G)$ satisfies: $T(i)$ is Suslin.*

Proof. Although the proof is classical, I shall give it since I shall refer to it later. T_a is clearly ω-closed. Let A be a maximal antichain in T_a. Let X be the L-least such that:

$$X \prec L_{\omega_3} \ \& \ \{A\} \cup \omega_1 \subset X \text{ and if } \lambda = \omega_2 \cap X, \text{ then } \lambda \in \omega_2 \text{ and } \mathrm{cf}(\lambda) = \omega_1$$

(clearly such an X exists). Let $\Pi : L_\beta \xrightarrow{\simeq} X$, so $\lambda = \omega_2 \cap X = \Pi^{-1}(\omega_2)$. Let g be a function from $\omega - a$ into ω_1 and

$$D_g = \{(\alpha, f) \in F_\lambda \mid (f_n(g(n)))_{n \in \omega - a} \text{ is compatible with some element of } A\}.$$

It is easy to see that D_g is a dense subset of F_λ and that $D_g \in L_{\eta_\lambda}$ (since $\eta_\lambda > \beta$ because $L_\beta \models \lambda = \omega_2$ whereas $L_{\eta_\lambda} \models \bar{\bar{\lambda}} = \omega_1$). It follows easily that $A = A \cap \lambda$ and so $\bar{\bar{A}} \leqslant \omega_1$.

Assume now $i \in a$, set $b = a - \{i\}$; then $T_b = T_a * T(i)$ (where $*$ means the iteration of forcing) satisfies the ω_2-chain condition and so $T(i)$ satisfies in $L(G)$ the ω_2-chain condition. $\square$

1.2. Coding the branches

For $n \in \omega$ and B_n a branch in $T(n)$ (in some generic extension of L). I shall define a set Q_n of conditions to code B_n by a subset of ω_1. It is a slight modification of the coding by use of almost disjoint sets and one of the basic idea of proof of the theorem.

1.2.1

Let $\alpha < \beta < \omega_1$ be such that: $L_\beta \models \alpha = \omega_1 + \mathrm{ZF}^- + \forall x\, (\bar{\bar{x}} \leqslant \omega_1) + \forall \lambda\, (\mathrm{cf}(\lambda) = \alpha \Rightarrow \eta_\lambda$ exists) (I shall denote $\Theta(\alpha, \beta)$ this property). I can construct in L_β the trees $T(n)$ exactly as I construct in L_{ω_2} the right ones. I shall denote $T_\beta(n)$ these trees.

Claim 1. Assume $\Theta(\alpha, \beta)$ and $\Theta(\alpha, \bar{\beta})$ and $\beta \leqslant \bar{\beta}$, then

$$T_\beta(n) = T_{\bar{\beta}}(n) \cap L_\beta.$$

This is an easy consequence of:

Claim 2. For $\gamma < \beta$:

(1) if $L_{\bar{\beta}} \models \mathrm{cf}(\gamma) = \omega$, then $L_\beta \models \mathrm{cf}(\gamma) = \omega$;

(2) if $f \in L_{\bar{\beta}}\, f : \omega \xrightarrow{\text{inj.}} T(n) \restriction \gamma$, then $f \in L_\beta$.

Proof. (1) Let $f : \omega \xrightarrow{\text{cof.,inj.}} \gamma\, f \in L_{\bar{\beta}}$; $g \in L_\beta\, g : \gamma \xrightarrow{\text{bij.}} \alpha$, then $g_0 f \in L_{\bar{\beta}}$ and $g_0 f : \omega \to \alpha$ so $g_0 f \in L_\alpha \subset L_\beta$ so $f = ((g_0 f)_0^{-1} g)^{-1} \in L_\beta$.

(2) let $g \in L_\beta g : T(n) \restriction \gamma \xrightarrow{\text{bij.}} \alpha$; then $g_0 f \in L_\alpha$ and so $f \in L_\beta$. $\square$

1.2.2

Fix n; let B_n be a branch in $T(n)$. I define the forcing Q_n in the following way. (I shall omit the subscript n when not necessary.)

Claim 1. There is a definable function $S: T \to P(\omega_1)$ such that:

(1) if $x \neq y$ $\overline{S(x) \cap S(y)} \leq \omega$;

(2) if $\Pi: L_\beta \to L_{\omega_2}$ is an elementary embedding such that $\pi(\alpha) = \omega_1$, then $\Pi(S(x) \cap \alpha) = S(\Pi(x))$.

Proof. Easy.

A condition in Q_n is a pair $(p, p^\cdot)$ such that:

(1) $p: |p| < \omega_1 \to 2$ such that for $\alpha \leq |p|$ and β if $\Theta(\alpha, \beta)$ and $L_\beta(p \restriction \alpha) \models \mathrm{ZF}^- + \alpha = \omega_1 = \omega_1^L$, then: $\{x \in T_\beta \mid S(x) \cap \tilde{p} \cap \alpha \cap E$ is bounded in $\alpha\}$ is a cofinal branch in T_β; (where $\tilde{p} = \{\gamma < |p| \mid p(\gamma) = 1\}$ and $E =$ the even ordinals $= \{\lambda + 2n \mid \lambda$ limit; $n \in \omega\}$.

(2) $\dot{p} \subset B \times \omega_2$; $\bar{\bar{p}}^\cdot \leq \omega$ and for $(x, \eta) \in p^\cdot$ $(S(x) - \eta) \cap \tilde{p} = \emptyset$. Set $(p, p^\cdot) \leq (q, q^\cdot)$ if $p \supset q$ and $p^\cdot \supset q^\cdot$.

Note. So Q is the usual forcing to code B by a subset of ω_1, except the fact that I ensure that this subset will have enough properties when later I shall use a condensation argument.

Lemma 1. *Let G be Q generic over $L(B)$; then there is a subset A of ω_1 such that:*

(1) $L(B)(G) = L(A)$;

(2) *for $x \in T$, $x \in B$ iff $S(x) \cap A \cap E$ is bounded in ω_1;*

(3) *if $\Theta(\alpha, \beta)$ and $L_\beta(A \cap \alpha) \models \mathrm{ZF}^- + \alpha = \omega_1$, then $\{x \in T_\beta \mid S(x) \cap A \cap E$ is bounded in $\alpha\}$ is a cofinal branch in T_β.*

Proof. The only non usual fact is that for any condition $(p, p^\cdot)$ and $|p| \leq \gamma < \omega_1$ there is a $q: \gamma \to 2$ such that $(q, q^\cdot)$ is a condition stronger than $(p, p^\cdot)$ (it is for that reason, I code B only on the even ordinals (to keep place for other things). Also note that since — later — I shall work with the Q_n all together, I have in fact to use a partition of ω_1 into ω parts).

To do that it is enough to put on the odd ordinals between $|p|$ and $|p| + \omega$ an order of type γ so that there are no $\alpha \in]|p|, \gamma]$ and β such that $L_\beta(q \restriction \alpha) \models \mathrm{ZF}^- + \alpha = \omega_1$. $\square$

The following is crucial:

Lemma 2. *Q is ω-distributive in $L(B)$.*

Proof. Q is equivalent to the two steps forcing.

(1) Add a subset A_0 of $\omega_1 \cap E$ to code B in the usual way. This is trivially distributive.

(2) The forcing $\bar{Q}$ defined as follows: a condition is a function $p: |p| \cap \omega_2 - E \to 2$ such that $|p| < \omega_1$ and for $\alpha \leq |p|$ and β, if $\Theta(\alpha, \beta)$ and $L_\beta(A_0 \cap \alpha, p \upharpoonright \alpha) \models \mathrm{ZF}^- + \alpha = \omega_1$, then $\{x \in T_\beta \mid S(x) \cap A_0 \cap \alpha$ is bounded in $\alpha\}$ is a cofinal branch in T_β.

It is clearly enough to show:

Claim 2. $\bar{Q}$ is ω-distributive in $L(B)(A_0) = L(A_0)$.

Proof. Let $(\Delta_i)_{i<\omega}$ be a sequence of strongly dense subsets of $\bar{Q}$ and $p_0 \in \bar{Q}$. Define $(X_i)_i$ by:

$$X_0 = \text{the smallest } X \prec L_{\omega_2}(A_0) \quad \text{such that} \quad p_0, (\Delta_i)_{i<\omega} \in X,$$

$$X_{i+1} = \text{the smallest } X \prec L_{\omega_2}(A_0) \quad \text{s.t. } \{X_i\} \cup X_i \subset X,$$

$$X_\omega = \bigcup_{i<\omega} X_i.$$

Let

$$\sigma_i : L_{\beta_i}(A_0 \cap \alpha_i) \xrightarrow{\simeq} X_i \quad \text{for } i \leq \omega.$$

Define $(p_i)_i$ by: $p_{i+1} =$ the least $p \leq p_i$ such that $p \in \Delta_i$ and $|p| \geq \alpha_i$. It is enough to show that $p = \bigcup_{i<\omega} p_i$ is a condition. It is easy to see that for $i < \omega$ $|p_i| < \alpha_i$ and so $|p| = \alpha_\omega$; so it remains to show that for $\beta > \alpha_\omega$ if $(*)$: $\Theta(\alpha_\omega, \beta)$ and $L_\beta(A_0 \cap \alpha_\omega, p) \models \mathrm{ZF}^- + \alpha_\omega = \omega_1$, then $\{x \in T_\beta \mid S(x) \cap A_0 \cap \alpha_\omega$ is bounded in $\alpha_\omega\}$ is a cofinal branch in T_β.

For $\beta \leq \beta_\omega$ this is trivial since $\sigma_\omega: L_{\beta_\omega}(A_0 \cap \alpha_\omega) \to L_{\omega_2}(A_0)$ is an elementary embedding and A_0 really codes a branch in T (it also uses Claim 1 in Section 1.2.1 and Claim 1 in Section 1.2.2).

For $\beta > \beta_\omega$ there is nothing to prove since $(*)$ does not occur: In $L_\beta(A_0 \cap \alpha_\omega)$ we can define the sequence $(\alpha_i)_{i<\omega}$ using $L_{\beta_\omega}(A_0 \cap \alpha_\omega)$ instead of $L_{\omega_2}(A_0)$ so $L_\beta(A_0 \cap \alpha_\omega) \models \bar{\bar{\alpha}}_\omega = \omega$. $\square$

1.3.

For $n \in \omega$ define $P_n = T(n) * Q_n$, i.e. P_n is the forcing that adds a branch in $T(n)$ and then codes it 'nicely' by a subset of ω_1.

Lemma 1. *Let $P = \prod_{n\in\omega} P_n$; then P is ω-distributive and satisfies the ω_2-chain condition.*

Proof. It is clear from the previous results. $\square$

Let M_1 be a P generic extension of M_0. New work in M_1.

Lemma 2. *There is a sequence $(A_n)_{n<\omega}$ of subsets of ω_1 such that M_1 satisfies:*

$$V = L((A_n)_{n\in\omega}).$$

I shall now define the last forcing C that gives the real a that will be Π^1_2 singleton; a will code (by use of almost disjoint sets) the A_n in such a way that $A_{2n}\in L(a)$ iff $a(n)=1$ and $A_{2n+1}\in L(a)$ iff $a(n)=0$. It will be shown that in $L(a)$ (as in some extension of this model) $T(2n)$ (resp. $T(2n+1)$) is Suslin iff $a(n)=0$ (resp. $=1$).

For $s\in{}^{<\omega}2$ define C_s by: r is a condition iff r is a pair (r_0, r_1) such that:

(1) r_0: $|r|<\omega\to 2$; $r_0\supset s$;
(2) $\bar{r}_1<\omega$ and

$$r_1\subset \omega\times\{(\alpha, 2i)\mid r_0(i)=1 \text{ and } \alpha\in A_{2i}\}$$
$$\cup\,\omega\times\{(\alpha, 2i+1)\mid r_0(i)=0 \text{ and } \alpha\in A_{2i+1}\};$$

(3) if $(n, \beta)\in r_1$, then:

$$(S(\beta)-n)\cap\tilde{r}_0=\emptyset;$$

where $\tilde{r}_0=\{i\mid r_0(i)=1\}$ and $\beta\to S(\beta)$ is some nice function giving almost disjoint subsets of ω.

Lemma 3. *For $s\in{}^{<\omega}2$, C_s satisfies the ω_1 chain condition.*

Let M_2 be a $C_\emptyset$ generic extension of M_1.

Lemma 4. *There is a real a in M_2 such that, setting*

$$D_n=\{\alpha\mid S((\alpha, n))\cap\tilde{a} \text{ is finite}\},$$

then:

$$\text{if } n=2i, \text{ then: } a(i)=1\Rightarrow D_n=A_n \text{ and } a(i)=0\Rightarrow D_n=\emptyset,$$

$$\text{if } n=2i+1, \text{ then: } a(i)=1\Rightarrow D_n=\emptyset \text{ and } a(i)=0\Rightarrow D_n=A_n.$$

Lemma 5. *Let $M_3=L(a)$ (M_3 is strictly included in M_2); then M_3 satisfies: $T(2i)$ is Suslin iff $a(i)=0$ and $T(2i+1)$ is Suslin iff $a(i)=1$.*

Proof. The part 'if' is trivial since if $a(i)=1$ (resp. 0), then A_{2i} (resp. A_{2i+1}) and so B_{2i} (resp. B_{2i+1}) is in $L(a)$.

For the 'only if' direction I first need:

Claim 1. M_2 is generic over M_0 by the following set of conditions:

> $F=P*C_\emptyset=\{(x_n)_n, (p_n, p_n^{\cdot})_n, (r_0, r_1)\mid$ where $x_n\in T(n)$, $(p_n, p_n^{\cdot})$ satisfies the definition of Q_n where B_n is replaced by $\{y\in T(n)\mid y\leq x_n\}$ and $(r_0; r_1)$ satisfies the definition of $C_\emptyset$ where A_j is replaced by $\tilde{p}_j\}$.

Proof. Trivial. □

Lemma 6. *Let $f=(p,(r_0,r_1))\in F$; then $F_f=\{g\in F \mid g\leq f\}$ is isomorphic to the product $F'\times F''$ where*

$$F'=\Big[\prod_{\substack{r_0(n)=1\\ \text{or}\\ n\geq|r_0|}} P_{2n}\times\prod_{\substack{r_0(n)=0\\ \text{or}\\ n\geq|r_0|}} P_{2n+1}\Big]*C_{r_0};$$

$$F''=\prod_{r_0(n)=0} P_{2n}\times\prod_{r_0(n)=1} P_{2n+1}.$$

Proof. Clear. □

Now suppose Lemma 5 is false; there is an i and $s=(p,(r_0,r_1))$ in the generic such that $r_0(i)=0$ and $s\Vdash T(2i)$ is not Suslin in $L(a)$. Using Lemma 6, write the generic subset of $F=(H',H'')$; it is clear that a is in $L(H')$, so it remains to show:

Claim 2. $T(2i)$ is Suslin in $L(H')$.

Proof. It is easy to see that

$$F'\simeq\Big(\prod_{\substack{r_0(n)=1\\ \text{or}\\ n\geq|r_0|}} Q_{2n}\times\prod_{\substack{r_0(n)=0\\ \text{or}\\ n\geq|r_0|}} Q_{2n+1}\Big)*C_{r_0}.$$

(This comes from the fact that forcing with $T(n)$ does not add subsets of ω_1.) □

It is then enough to show that for each step $T(2i)$ remains Suslin. For the first one, it is exactly the lemma in Section 1.1. For the other ones it comes from the following:

Lemma 7. *Let N be a model of* ZF *and T be in N a Suslin tree of height ω_2; let P be a notion of forcing with the strong ω_2-chain condition; then if $\bar N$ is a P generic extension of N, $\bar N$ satisfies: T is Suslin.*

Proof. Let $A\in\bar N$ be an antichain in T; let $\mathring{A}$ be a name for A and p in the generic such that: $p\Vdash\mathring{A}$ is an antichain in $\check{T}$.

Let $B=\{x\in T\mid \exists\, q\leq p,\, q\Vdash\check{x}\in\mathring{A}\}$. Then $B\in N$ and $A\subset B$; so it suffices to show that $\bar{\bar{B}}\leq\omega_1$ (in N). Now work in N. For each $x\in B$, pick $q_x\leq p$ such that $q_x\Vdash\check{x}\in\mathring{A}$; note that if $x\neq x'$ and q_x and $q_{x'}$ are compatible, then x and x' are uncompatible. Suppose B has cardinality ω_2; let $C=\{q_x\mid x\in B\}$.

If C has cardinality ω_1, then for some q the set $\{x\mid q\Vdash\check{x}\in\mathring{A}\}$ is an antichain of T of cardinality ω_2; this is impossible.

So C contains (by the strong ω_2-chain condition) a subset D of pairwise compatible conditions, $\bar{\bar{D}} = \omega_2$; but then the set $\{x \mid q_x \in D\}$ is an antichain of T of cardinality ω_2; and the proof is complete. □

1.4.

Let $\varphi(x)$ be the formula:

$\forall\alpha < \beta < \omega_1$ if $(L_\beta \models \mathrm{ZF}^- + \alpha = \omega_1 + \forall x\,(\bar{\bar{x}} \leqslant \omega_1) + \forall\lambda\,(\mathrm{cf}(\lambda) = \alpha \Rightarrow \eta_\lambda$ exists) and $L_\beta(x) \models \mathrm{ZF}^- + \alpha = \omega_1$, then: assume you decode x into ω subsets of α, which themselves are decoded into subsets B_n of $T_\beta(n)$, then:

If $x(i) = 1$, then B_{2i} is a cofinal branch in $T_\beta(2i)$.

If $x(i) = 0$, then B_{2i+1} is a cofinal branch in $T_\beta(2i+1)$.

Claim. φ is equivalent to a Π^1_2 formula.

Lemma 1. (1) $L(a) \models \varphi(a)$.

(2) *For any real x, if* $L(x) \models \varphi(x)$, *then*:

If $x(i) = 1$, $T(2i)$ *is not Suslin in* $L(x)$.

If $x(i) = 0$, $T(2i+1)$ *is not Suslin in* $L(x)$.

Proof. (1) Assume α, β satisfy the hypothesis in φ; when a is decoded the sequence $(A_i \cap \alpha)_{i<\omega}$ is obtained and then the conclusion follows from the Lemma 1 in Section 1.2.2.

(2) It follows immediately from the Lowenheim–Skolem theorem. □

Now Theorem 2 follows easily from all the previous results.

1.5.

The idea of the proof of Theorem 1 is simply: Do the same thing with the trees $T(n)$ not only on ω_2 but on all the cardinals, and choose a real a that codes (by Jensen's method, with enough condensation properties) branches in the $T(n)$ in such a way that a can be recovered by looking whether $T(n)$ is Suslin or not. Doing that is only a slight modification of Jensen's proof; the important one comes from the following:

One of the important fact in the proof of Theorem 2 is the lemma which essentially says that for $n \neq m$ $T(n)$ remains Suslin when forcing with P_m; this comes first from the property of the sequence $(T(n))_n$ (that is easy to extend to greater cardinals) and secondly from the fact that Q_m is small in view of $T(n)$ (the strong chain condition).

In the general case, when P_m will be a class of conditions, P_m will not be 'small' in view of $T(n)$; of course P_m can be cut into its small and big part and one has the lemma:

Lemma. *Let M be a model of* ZF, α *a cardinal in M and T a Suslin tree of height*

α^+ in M; let P be a notion of forcing which is α closed (i.e.: every decreasing chain of conditions of length α has a lower bound). Then if N is a P generic extension of M, then N satisfies: T is Suslin.

Unfortunately the big part of P_m only is α distributive and this is not sufficient to preserve the Suslinity of a tree (since forcing with the tree itself is distributive!). Happily there is a solution for that: the idea is the following: Work with the $T(n)$ defined in Section 1.1; recall that the level λ for $\mathrm{cf}(\lambda)=\omega_1$ has been defined by use of forcing over some L_{η_λ}. Assume C is some forcing which preserves the cardinals and adds a subset D of ω_2 such that for each λ, $\mathrm{cf}(\lambda)=\omega_1$, $D\cap\lambda\in L_{\eta_\lambda}$; it follows easily from the proof of the lemma in Section 1.1 that the $T(n)$ remain Suslin in $L(D)$.

The essential — but not so difficult! — fact that has to be examined in the iteration is:

Claim. In the Jensen's conditions, the level of constructibility of the conditions can be controlled.

2. The iteration

2.1.

Definition 1. Let α be a cardinal; I define $\chi_n(\alpha)$ by:

$$\chi_0(\alpha)=\alpha,$$

$$\chi_{n+1}(\alpha)=\chi_n(\alpha)^+.$$

Definition 2. Let α be a cardinal; for $\xi\in[\alpha,\alpha^+[$ I define by induction the ordinals μ_ξ^i $(i\leq\omega)$ as follows:

$$\mu_\xi^0=\sup(\alpha\cup\{\mu_\eta^\omega\mid\eta\in[\alpha,\xi[\});$$

$\mu_\xi^{i+1}=$ the least $\mu>\mu_\xi^i$ such that:

(i) $L_\mu\vDash\mathrm{ZF}^-+\forall x\ \bar{\bar{x}}\leq\alpha$,

(ii) if α is a successor cardinal $\mathrm{cf}(\mu)=\alpha$;

$$\mu_\xi^\omega=\sup(\mu_\xi^i\mid i<\omega).$$

Definition 3. Let α be a *limit* cardinal and $n\in\omega$, I define *one* Suslin tree $T(n,\alpha)$ of height $\chi_{n+2}(\alpha)$ in the same way the sequence $T(n)$ has been defined in Section 1. The levels of the tree are defined by induction; for limit λ such that $\mathrm{cf}(\lambda)=\chi_{n+1}(\alpha)$ the level is defined by forcing over $L_{\mu_\lambda^1}$.

Note. (1) If β is an ordinal such that $L_\beta\vDash\mathrm{ZF}^-+$there is a greatest cardinal and

$$\forall\xi<\beta\ \forall i\leq\omega\ \mu_\xi^i<\beta;$$

then the trees $T(n,\gamma)$ can be defined in L_β — by the same definition as the right ones — for any γ such that L_β satisfies: γ is a limit cardinal such that $\chi_{n+1}(\gamma)$ exists.

(2) If $\beta \leqslant \bar{\beta}$ satisfy the hypothesis of (1) and L_β and $L_{\bar{\beta}}$ have the same cardinals, then the $T_\beta(n, \gamma)$ and the $T_{\bar{\beta}}(n, \gamma)$ are the same if $\chi_{n+1}(\gamma)$ is not the greatest cardinal; and if L_β and $L_{\bar{\beta}}$ satisfy: $\chi_{n+1}(\gamma)$ is the greatest cardinal, then:

$$T_\beta(n, \gamma) = T_{\bar{\beta}}(n, \gamma) \cap L_\beta.$$

(3) Since now I shall work on a class of cardinals one tree by level will be enough and simpler.

(4) If $|x|$ is the level of x in T, note that $x \in L_{\mu^2_{|x|}}$ for any x and that:

$$\forall x \in T(n, \alpha)\,(\mathrm{cf}(|x|) < \chi_{n+1}(\alpha) \Rightarrow x \in L_{\mu^1_{|x|}}).$$

(This will be often used without mention.)

2.2.

Now I am going to define, for each n, a forcing $P(n)$ which looks like the final forcing P_ω of Jensen. Fix n up to the end of this section (the subscript n will be often omitted). In the following I shall also follow Jensen's notations (see [1]).

Definition 1. Let α be a cardinal $\geqslant \omega$; define S_α as follows:

(I) If α is not $\chi_{n+1}(\gamma)$ for some limit cardinal γ, then $s \in S_\alpha$ iff $s : [\alpha, |s|[\to 2$ such that $|s| < \alpha^+$ and $\forall \xi \leqslant |s|$:

(1) $s \upharpoonright \xi \in L_{\mu^1_\xi}$;

(2) $\forall \beta\, L_\beta(s \upharpoonright \xi)$ satisfies: "if $\Theta(\alpha, \xi, s \upharpoonright \xi)$, then $\Gamma(\alpha, \xi, s \upharpoonright \xi)$" where Θ is "$\mathrm{ZF}^- + \xi = \alpha^+ +$ there is a greatest cardinal (not necessarily ξ!) $+ \forall i \leqslant \omega\, \forall \nu\, \mu^i_\nu$ exists" and Γ is: "when $s \upharpoonright \xi$ is decoded by Jensen's method, a cofinal branch is founded in each $T_\beta(n, \gamma)$ such that $L_\beta \vDash \gamma$ is a limit cardinal such that $\chi_{n+1}(\gamma)$ exists".

(II) If α is $\chi_{n+1}(\gamma)$ for some limit γ, then (1) is replaced by:

(1′) $s \upharpoonright Z_0$ is a branch in $T(n, \gamma)$ (recall T is a binary tree);

(1″) $s \upharpoonright \xi \in L_{\mu^1_\xi}(s \upharpoonright Z_0 \cap \xi)$;

where as in [1] $Z_\nu = \{(\eta, \nu) \mid \eta \in \mathrm{Ord}\}$ and $(\cdot, \cdot)$ is the Gödel pairing function.

Note. In [1] for $s \in S_\alpha$, ordinals μ^i_s are defined.

Claim. The μ^i_s defined in [1] are the $\mu^i_{|s|}$ of my Definition 2.

Proof. In the case (I) this is trivial since $s \in L_{\mu^1_s}$.

In the case (II) I only have to show that $b = L_{\mu^1_{|s|}}(s \upharpoonright Z_0)$ satisfies the conditions $a \to d$ of Jensen's definition of the μ^1_s in the case $\mathrm{cf}(|s|) = \chi_{n+1}(\gamma)$ (by Note 4 in Definition 3); but then $s \upharpoonright Z_0$ is (by the construction of T) generic over $L_{\mu^1_{|s|}}$ and the result follows easily. □

Definition 2. For α a cardinal and $s \in S_{\alpha^+}$, R_s is the set of pairs $(r, r^\cdot)$ such that:

(i) $r \in S_\alpha$;

(ii) $(r, r^\cdot)$ codes s as in [1].

Note. Here there is no $A\cap\alpha^+$: so the generic will be a Cohen in Z_0 if α is not $\chi_{n+1}(\gamma)$ for some limit cardinal γ and a branch in $T(n,\gamma)$ in the other case. Define also the $\mathscr{A}^i_s$ ($i\leqslant\omega$; $s\in S_\alpha$) as in [1].

Claim. (1) For $i\neq 1$ $\mathscr{A}^i_s=L_{\mu^i_{|s|}}$.

(2) For $i=1$ $\mathscr{A}^1_s=L_{\mu^1_{|s|}}$ if α is not $\chi_{n+1}(\gamma)$ for some limit γ and $\mathscr{A}^1_s=L_{\mu^1_{|s|}}(s\restriction Z_0)$ in the other case.

Lemma 1. *R_s is α distributive in $\mathscr{A}^1_s$.*

Proof. Forcing with R_s is equivalent to the three following forcing operations

(1) R_0 to code s on Z_1 and so add D_0 a subset of $[\alpha,\alpha^+[\cap Z_1$ in such a way that:

$$\forall\,\xi\in[\alpha,\alpha^+[\ \ D_0\cap\xi\in L_{\mu^1_\xi}.$$

Claim. R_0 is α distributive in $\mathscr{A}^1_s$.

Proof. Assume $(\Delta_i)_{i<\alpha}$ is a sequence of strongly dense subsets of R_0, $(\Delta_i)_{i<\alpha}\in\mathscr{A}^1_s$ and $p_0\in R_0$; set $b=\mathscr{A}^2_s=L_{\mu^2_{|s|}}$.

Define a sequence $(X_i)_{i\leqslant\alpha}$ as follows:

$$X_0=\text{the least } X<b \quad\text{such that } \alpha\cup\{p_0,(\Delta_i)_i\}\subset X;$$

$$X_{i+1}=\text{the least } X<b \quad\text{such that } X_i\cup\{X_i\}\subset X;$$

$$X_\lambda=\bigcup_{i<\lambda}X_i \quad\text{for limit } \lambda.$$

Set $\sigma_i:b_i=L_{\beta_i}\xrightarrow{\sim}X_i$ and $\alpha_i=X_i\cap\alpha^+=\sigma_i^{-1}(\alpha^+)$.

Define a sequence $(p_i)_{i\leqslant\alpha}$ of conditions by:

$$p_{i+1}=\text{the least } p\leqslant p_i \quad\text{such that } p\in\Delta_i \text{ and } |p|\geqslant\alpha_i$$

(I shall prove later — in Section 2.4 — that such a p exists);

$$p_\lambda=\bigcup_{i<\lambda}p_i \quad\text{for limit } \lambda \text{ if } \bigcup p_i \text{ is a condition, undefined if not.}$$

As in [1] the only problem is to show that for limit λ $p=\bigcup_{i<\lambda}p_i$ is a condition. And for that I must show that $p\in L_{\mu^1_{\alpha_\lambda}}$. But as in [1] it is easy to see that $p\in L_{\beta_\lambda+1}$ (p can be defined in $L_{\beta_\lambda+1}$ by use of b_λ instead of b), and since $\beta_\lambda<\mu^1_{\alpha_\lambda}$ we are done. We now work in $\mathscr{A}^1_s(D_0)=L_{\mu^1_{|s|}}(D_0)$. □

(2) The second step is (if α is $\chi_{n+1}(\gamma)$ for some limit γ): add a branch B in $T(n,\gamma)$; this is trivially α distributive since T remains Suslin in $L_{\mu^1_s}(D_0)$.

(3) The third has to ensure the property (2) in the definition of S_α. A condition is a function r:

$$[\alpha, |r|[\cap \bigcup_{\beta \in I} Z_\nu \to 2$$

such that $|r| < \alpha^+$ and $I = [2, \alpha^+[$ if (2) has occurred (resp. $\{0\} \cup [2, \alpha^+[$ if not) and $\forall \xi \in [\alpha, |r|]$

(i) $r \restriction \xi \in L_{\mu^1_\xi}(B \cap \xi)$ (resp. $L_{\mu^1_\xi}$)

(ii) $\forall \beta L_\beta(D_0 \cap \xi, B \cap \xi, r \restriction \xi)$ (resp. $L_\beta(D_0 \cap \xi, r \cap \xi)$) satisfies: "if $\Theta(\alpha, \xi, (D_0, B, r) \restriction \xi)$, then $\Gamma(-----)$" (resp. "if $\Theta(\alpha, \xi, (D_0, r) \restriction \xi)$, then $\Gamma(-----)$"). The proof that this forcing is α distributive in $L_{\mu^1_s}(D_0, B)$ (resp. $L_{\mu^1_s}(D_0)$) uses essentially the same argument as in Lemma 2 in Section 1.2, but since some care is needed, I shall repeat it.

I shall develop the argument in the first case (when B occurs). It is the same in the other case.

Begin as in the proof for R_0, using $b = L_{\mu^1_s}(D_0, B)$ and define X_i, α_i, p_i ($i \leq \alpha$) in the same way; as usual I have to prove (1″) and (2) in the definition of S_α only for $\xi = \alpha_\lambda = |p|$ when λ is limit and $p = \bigcup_{i<\lambda} p_i$.

(1″) is as for R_0: it is easy to see that

$$p \in L_{\beta_\lambda+1}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda) \subset L_{\mu^1_{\alpha_\lambda}}(B \cap \alpha_\lambda).$$

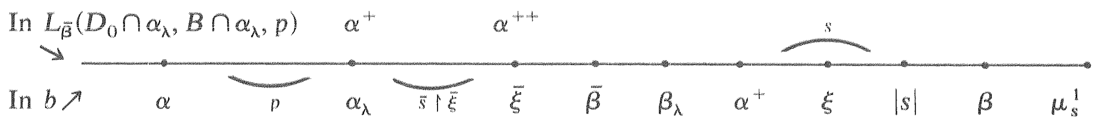
I now prove (2): choose $\bar\beta$ such that $L_{\bar\beta}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda, p) \models \Theta$.

Since $(\alpha_i)_{i<\lambda} \in L_{\beta_\lambda+1}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda)$, $\bar\beta$ is not greater than β_λ set $\bar s = \sigma_\lambda^{-1}(s)$ (recall $s \in b$).

(A) Assume first there is a $\bar\xi$ such that $L_{\bar\beta} \models \bar\xi = \alpha^{++}$, then $\bar\beta < \beta_\lambda$ (since $\sigma_\lambda : L_{\beta_\lambda} \xrightarrow{\sim} L_{\mu^1_s}$ and the definition of μ^i_s ensures that $L_{\mu^1_s} \models \alpha^{++}$ does not exist).

Claim. $\bar\xi \leq |\bar s|$.

If $|\bar s| < \bar\xi$, then $\mu^1_{\bar s} < \bar\xi < \bar\beta < \beta_\lambda$ and so (by σ_λ) μ^1_s exists in b; a contradiction. Set $\xi = \sigma_\lambda(\bar\xi)$ (so $\xi \leq |s|$) and $\beta = \sigma_\lambda(\bar\beta)$; the picture is:



Now work in $L_{\bar\beta}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda, p)$: $B \cap \alpha_\lambda$ really is a branch; if you decode you first find $\bar s$ and $L_{\bar\beta}(\bar s \restriction \bar\xi)$ satisfies $\Theta(\alpha_\lambda, \bar\xi, \bar s \restriction \bar\xi)$ so (by σ_λ) $L_\beta(s \restriction \xi)$ satisfies $\Theta(\alpha^+, \xi, s \restriction \xi)$; since $s \in S_{\alpha^+}$ and (by σ_λ) $L_{\bar\beta}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda, \bar s \restriction \bar\xi)$ satisfies Γ; and this is exactly what we need.

(B) Assume now $L_{\bar\beta} \models \alpha_\lambda$ is the greatest cardinal.

Claim. $\bar\beta \leq \bar s$.

If $|\bar s| = \bar\xi < \bar\beta$, then by the definition of Θ, $\mu^1_{\bar s} < \bar\beta \leq \beta_\lambda$ and so by σ_λ, μ^1_s exists in b; a contradiction. So in $L_{\bar\beta}$ $(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda, p)$ we can recover $\bar s \restriction \bar\beta$ and so by the elementarity of $\sigma_\lambda \bar{s} \restriction \bar{\beta} \cap Z_0$ is a branch in the tree (if it has to be one) (note that this does not occur when (B) appears — since that last tree has height α_λ — however, I give the argument, since when (B) does not appear, this case can occur). This achieves the proof of the lemma. □

Defining S_α^+ and R^D as in [1] (see Definition 2, p. 37) we have:

Lemma 2. *R^D satisfies the α^{++}-chain condition in $\mathscr{A}_D$.*

Proof. Each step of the iteration satisfies it — and even the strong chain condition, except for adding a branch in the tree. □

2.3.

The other definitions and lemmas used to define what is called P_ω in [1] are exactly as in [1], except one point: It is more convenient to define $\rho^i_{s\tau}$ to be a code for $\mathscr{A}^i_{s,\chi_{n+2}(\tau)}$ (instead of $\mathscr{A}^i_{\sigma\tau^+}$ in [1]). That does not modify the proof in [1] but for me it ensures that there is no intrusion of the coding at limit cardinals with the branches. Now what I call $P(n)$ (remind n was fixed) is what it called P_ω in [1].

I have to show the lemmas that say:

(1) a condition can be arbitrarily extended;

(2) $P_\tau(n)$ is τ-distributive.

I shall prove that later and now show how to conclude. At the moment the following will be clear.

Lemma 1. *Let N be a $P(n)$ generic extension of M_0; then there is a subset A_n of ω_1 such that N satisfies:*

(1) $\mathrm{ZF} + V = L(A_n)$.

(2) $\forall \alpha < \omega_1 \, \forall \beta$ *if* $L_\beta(A_n \cap \alpha) \models \mathrm{ZF}^- +$ *there is a greatest cardinal* $+ \alpha = \omega_1 = \omega_1^L$, *then if you decode $A_n \cap \alpha$ by Jensen's method you find a cofinal branch in each $T_\beta(n, \gamma)$ such that $L_\beta \models \gamma$ is a limit cardinal and $\chi_{n+1}(\gamma)$ exists.*

Let M_1 be a $\prod_{n\in\omega} P(n)$ generic extension of M_0. Define the forcing C_s as in Section 1. Let M_2 be a $C_\emptyset$ generic extension of M_1. This gives the real a for which I must prove.

Lemma 2. *For all the limit γ, $L(a)$ satisfies:*

$T(2i, \gamma)$ is Suslin iff $a(i) = 0$,

$T(2i+1, \gamma)$ is Suslin iff $a(i) = 1$.

This follows — as in Section 1 — from Lemmas 3 to 5.

Lemma 3. *Let $f = (p(n))_{n\in\omega}, (r_0, r_1))$ be a condition in*

$$F = \prod_{n\in\omega} P(n) * C_\emptyset,$$

then $F_f = \{g \in F \mid g \leq f\}$ *is isomorphic to the product* $F' \times F''$ *where*

$$F' = \Big(\prod_{\substack{r_0(n)=1\\ \text{or}\\ n \geq |r_0|}} P(2n) \times \prod_{\substack{r_0(n)=0\\ \text{or}\\ n \geq |r_0|}} P(2n+1)\Big) * C_{r_0},$$

$$F'' = \Big(\prod_{r_0(n)=0} P(2n) \times \prod_{r_0(n)=1} P(2n+1)\Big).$$

Proof. Trivial. $\square$

Lemma 4. *Let* γ *be a limit cardinal,* a *be a finite subset of* ω *and* $n \in a$. *Let* N *be a* $\prod_{m \notin a} P(m)$ *generic extension of* M_0, *then* N *satisfies*: $T(n, \gamma)$ *is Suslin.*

Proof. Set $\alpha = \chi_{n+1}(\gamma)$; still using the notation in [1], there are subsets D_m $(m \notin a)$ of $[\alpha, \alpha^+[$such that: $P(m) = P_\alpha(m) * P_\omega^{D_m}(m)$.

Claim.

$$\prod_{m \notin a} P(m) \simeq \Big(\prod_{m \notin a} P_\alpha(m)\Big) * \Big(\prod_{m \notin a} P_\omega^{D_m}(m)\Big).$$

Proof. This follows easily from the fact that $\prod_{m \notin a} P_\alpha(m)$ is α distributive and α is a successor cardinal. $\square$

So $\prod_{m \notin a} P(m)$ is as a two step forcing:
The first one gives a model that satisfies:

$$V = L((D_m)_{m \notin a}) + \forall\, m \notin a\; D_m \subset [\alpha, \alpha^+[+ \forall\, \xi \in [\alpha, \alpha^+[\forall m\; D_m \upharpoonright \xi \in L_{\mu^1_\xi}.$$

So $T(n, \gamma)$ is Suslin in $L((D_m)_{m \notin a})$; this follows from the Lemma 6 below.

The second one satisfies the α^+ strong chain condition, so $T(n, \gamma)$ remains Suslin by Lemma 5 below. $\square$

Lemma 5. *Let* M *be a model of* ZF, α *a cardinal and* T *a Suslin tree of height* α^+; *let* P *be a notion of forcing that satisfies the* α^+ *strong chain condition and* N *be a* P *generic extension of* M; *then* N *satisfies*: T *is Suslin.*

Proof. As in Lemma 7 in Section 1.3. $\square$

Lemma 6. *Let* M *be a model of*: ZF $+ V = L(D)$, $D \subset \alpha^+$ *for some regular cardinal* α. *Let* T *be the Suslin tree of height* α^+ *defined in* L *as in Section* 1 (*by forcing over* $L_{\mu^1_\xi}$ *for* $\mathrm{cf}(\xi) = \alpha$). *Assume*: $\forall\, \xi < \alpha^+(\mathrm{cf}(\xi) = \alpha \to D \upharpoonright \xi \in L_{\mu^1_\xi})$. *Then* T *is Suslin in* M.

Proof. This is an easy corollary to the proof of the (generalized) lemma in Section 1.1. $\square$

Definition. Let φ be the formula: $\forall\, \alpha < \beta < \omega_1$ if $L_\beta(x) \models \mathrm{ZF}^- + \alpha = \omega_1 = \omega_1^L +$there is a greatest cardinal $+ \forall\, \xi\, \forall\, i < \omega\, \mu_\xi^i$ exists, then if you decode x by Jensen's method you find a cofinal branch in $T_\beta(n, \gamma)$ for each γ such that $L_\beta \models \gamma$ is a limit cardinal and for each n such that:

$$n = 2i \quad \text{and} \quad x(i) = 1$$

or

$$n = 2i + 1 \quad \text{and} \quad x(i) = 0.$$

Now the theorem follows easily from all the previous results; the argument for (3) in the theorem being: (by use of Lemma 5) you cannot kill all the $T(n, \gamma)$ by forcing with a set of conditions.

It remains now to prove that conditions can be arbitrarily extended and then distributivity.

2.4.

Lemma 1. *Let ψ be a formula; α a cardinal; let C be the following notion of forcing: a condition is a function $c : [\alpha, |c|[\to 2$ such that $|c| < \alpha^+$ and for $\xi \leq |c|$*

(1) $c \restriction \xi \in L_{\mu_\xi^1}$,

(2) $\forall \beta\, L_\beta(c \restriction \xi) \models (\mathrm{ZF}^- + \xi = \alpha^+ \Rightarrow \psi)$,

then for any condition c_0 and $\gamma \in [|c|, \alpha^+[$ there is a condition $c \leq c_0$ such that $|c| = \gamma$.

As a corrollary of this lemma we immediately have:

Lemma 2. *Any condition in R_s can be arbitrarily extended.*

Proof of Lemma 1. By induction on γ; for γ successor this is trivial, so let γ be a limit ordinal.

The proof of $\square_\alpha$ (see for instance [1, Section 6]) shows that there is a closed unbounded set D of $[\alpha, \gamma[$such that $|D| \leq \alpha$ (the order type of D) and for all $\nu \in D \cup \{\gamma\}$, $D \cap \nu \in L_{\mu_\nu^1}$.

Let $(\gamma_i)_{i<\lambda}$ be the monotone enumeration of D; I may assume $\gamma_0 = |c_0|$ and γ_i is a limit ordinal for $i < \lambda$.

For any ordinal $\beta = \xi + n$ where ξ is limit set $f(\beta) = \xi + 2n + 1$.

Define c_i $(i \leq \lambda)$ by induction as follows:

$c_{i+1} = c_i \cup \chi_E$ where χ_E is the characteristic function of the set $E = \{\gamma_i\} \cup \{f(\beta) \mid \beta \in c'\}$ where c' is the least c such that $c \subset [\gamma_i, \gamma_{i+1}[$ and $c_i \cup \chi_c$ is a condition.

$c_\eta = \bigcup_{i<\eta} c_i$ for limit η if $\bigcup c_i$ is a condition, undefined if not.

As usual it is enough to show that for limit η $c = \bigcup_{i<\eta} c_i$ is a condition and for that to see that (1) and (2) in the definition of C is true for $\xi = \gamma_\eta$. (2) is trivial since $D \cap \gamma_\eta$ can be recovered from c so there is no β such that

$$L_\beta(c) \models \gamma_\eta = \alpha^+.$$

(1) since $D \cap \gamma_\eta \in L_{\mu^1_{\gamma_\eta}}$ it follows easily that $c \in L_{\mu^1_{\gamma_\eta}}$. $\square$

The other lemma that has to be proved is lemma I.6 in [1] with the new condition that control the level of constructibility.

Definition 1. *Let α be a cardinal; define $\tilde{S}_\alpha$ to be the set of functions $s : [\alpha, |s|[\to 2$ such that $|s| < \alpha^+$ and for $\xi \leq |s|$ $s \restriction \xi \in L_{\mu^1_\xi}$.*

Definition 2. Let κ be a *limit* cardinal, A a subset of $[\kappa, \kappa^+[$ such that: $\forall \xi < \kappa^+$ $A \cap \xi \in L_{\mu^1_\xi}$; let I be the set of *successor* cardinals less than κ. For τ a cardinal less than κ define $\tilde{P}_\tau$ by:

A condition is a sequence $p = (p_\gamma)_{\gamma \in I \cap [\tau, \kappa[}$ such that:

(1) $\forall \gamma\, p_\gamma \in \tilde{S}_\gamma$.

(2) If $|p|$ is defined by: $|p| =$ the least ξ such that $p \in L_{\mu^1_\xi}$, then for $\xi < |p|$ there is a $\nu \in I$ such that: $\forall \gamma \in [\nu, \kappa[\cap I$

$$p_\gamma(\rho_{\xi\gamma}) = \begin{cases} 1 & \text{if } \xi \in A, \\ 0 & \text{if not} \end{cases}$$

(where the $\rho_{\xi\gamma}$ are defined as in [5]).

Lemma 3. *For any $p \in \tilde{P}_\tau$ and $\xi \geq |p|$ there is a $q \leq p$ such that $|q| = \xi$.*

The proof follows the proof in [1]: I shall need:

Lemma 4. *Let γ be a successor cardinal; $\xi \in [\gamma, \gamma^+[$ such that $\mathrm{cf}(\xi) < \gamma$; $\mu \in]\xi, \gamma^+[$ such that $\mathrm{cf}(\mu) = \gamma$ and $L_\mu \models \mathrm{ZF}^- + \forall x (\bar{\bar{x}} \leq \gamma)$; $X \subset \xi$ such that $X \in L$ and $\forall \eta < \xi$ $X \cap \eta \in L_\mu$. Then $X \in L_\mu$.*

Proof. Set $\lambda = \mathrm{cf}(\xi) < \gamma$; the proof follows immediately from the two claims.

Claim 1.

$$L_\mu \models \mathrm{cf}(\xi) = \lambda.$$

Let $f \in L$; $f : \lambda \to \xi$ be cofinal, injective; $g \in L_\mu$; $g : \xi \to \gamma$ bijective, then $g_0 f : \lambda \to \gamma$ injective, so since γ is regular $g_0 f \in L_\gamma \subset L_\mu$ and so $f \in L_\mu$.

Claim 2. Let $f \in L_\mu : \lambda \to \xi$ be cofinal, increasing and $x_i = X \cap f(i)$; then $(x_i)_{i<\lambda} \in L_\mu$.

Let $\varphi : \mu \to L_\mu$ be the canonical enumeration of L_μ; set $\eta_i = \varphi^{-1}(x_i)$; it is enough to show: $(\eta_i)_{i<\lambda} \in L_\mu$ but this follows by the same argument as in Claim 1 from $\mathrm{cf}(\mu) = \gamma > \lambda$.

Proof of Lemma 3. The successor case is as in [1] (I of course use Lemma 1 to extend p up to $\rho_{\xi\gamma}$).

For the limit case, I extend p using a closed unbounded subset of $[|p|, \xi[$ coming from a $\square$ sequence. As usual it is enough to prove that for a limit λ if $p = \bigcup_{i<\lambda} p^i$, then for $\gamma \in I\, p_\gamma \in L_{\mu^1_{p_\gamma}}$. It follows easily from Lemma 4 and the fact I may always assume that $\lambda < \gamma$ (if the order type of the c.u.b. is less than κ this is easy; if not, then κ is regular and the p^i are defined in such a way that $p^\lambda_\gamma = \bigcup_{i<\lambda} p^i_\gamma$ is a non trivial union only for $\gamma > \lambda$ (remind γ is a successor)).

Now the proof of the extendability of the conditions is exactly as in [1], using the previous lemmas and specially the Lemma 4.

The only (slight) different point is in Lemma 2.14.3 of [1] when the r_{ij}, $\tilde{q}_{ij}$ are defined. I must be more careful since for limit i $(r'_j)_{j<i}$ is not necessarily in $\mathscr{A}^1_{r'_i}$; so r_{i1} has to be defined in the following way:
Let ξ_i be the least such that $x_i \in L_{\mu^1_{\xi_i}}$ where $x_i = (p \restriction \beta_i; (r_{h0} \mid h \leqslant i); \tilde{q} \restriction \beta_i)$; define r_{i1} by: $r_{i0} \cup \{(\eta_i, 5)\} \cup r'$ where r' is such that:

(1) η_i is s.t. $L_{\mu^1_{\xi_i}} \models x_i$ has number η_i in the canonical well ordering of V;

(2) r' is the least $r \subset [|r_{i0}|, \mu_{\xi_i}[\cap Z_4$ such that $r_{i1} \in S_{\beta_i}$. It is then easy to see that

It is then easy to see that

$$r^*_i = \bigcup_{j<\omega} r_{ij} \in S_{\beta_i}$$

and the end of the lemma is the same. $\square$

2.5.

The distributivity is proved as in [1]. I shall only mention the points where something has to be changed or where some care is needed. Lemma 3.7 in [1] has to be replaced by:

Lemma. *Assume distributivity holds up to α; let $s \in S_{\alpha^+}$, $\tau \leqslant \alpha$ and let $D \subset [\tau, \alpha^+[$ be P^s_τ generic over $\mathscr{A}^1_s$ then:*

(1) *$s, D \in L_{\mu^1_s}(D_\tau)$;*

(2) *for $\gamma \in [\tau, \alpha]$ and $\xi \in [\gamma, \gamma^+[$ then*

(i) *$D_\gamma \restriction \xi \in L_{\mu^1_\xi}$ (resp. if $\gamma = \chi_{n+1}(\nu)$ for some limit cardinal ν $D_\gamma \cap Z_0$ is a branch in $T(n, \nu)$ and $D_\gamma \restriction \xi \in L_{\mu^1_\xi}(D_\gamma \cap Z_0 \cap \xi)$);*

(ii) *$\forall \beta\, L_\beta(D_\tau \cap \xi)$ satisfies: "if Θ, then Γ".*

Proof. As in [1]. $\square$

The only other point is in the proof of Lemma 3.22(a) in [1]. I have to verify the additional property of S_γ.

There are two things to look at: the usual facts in [1] and the branches in the trees. Moreover we have to show that $\prod_n P^s_\tau(n)$ is τ-distributive, not only that for each n $P^s_\tau(n)$ is τ-distributive, since the product of τ-distributive forcings is not necessarily τ-distributive.

What about branches?

Note that

$$\prod_n P^s_\tau(n) \simeq P^s_{\tau^+}(n) * \prod_n R^{D_n} \quad \text{for some generic sets } D_n.$$

With the first forcing there are no problems with the branches since all the trees involved in this forcing are τ-closed.

For the second one there are no problems too with the branches except for the n (if there is one) such that:

$$\tau = \chi_{n+1}(\beta) \quad \text{for some limit } \beta.$$

But $T(n, \beta)$ is Suslin in $L_{\mu^1_s}(D_m \mid m < \omega)$, since $L_{\mu^1_s}$ and $L_{\mu^1_s}(D_m \mid m < \omega)$ have the same subsets of τ^+ and it is well known that forcing with a Suslin tree of height τ^+ is τ distributive.

So I can 'forget' the branches and only have to look at the properties (1) and (2) in S_γ: We start the construction with $\mathscr{A}^2_s = L_{\mu_s}$ instead of $\mathscr{A}^1_s$ in [1].

The b appearing in the lemma is some $L_{\mu'}$ with $\mu' < \mu_{|p^\lambda_\gamma|}$ but since the sequence $(p^i)_{i<\lambda}$ is definable from b as it was defined from $\mathscr{A}^2_s$ it follows that $p^\lambda_\gamma \in L_{\mu_{|p^\lambda_\gamma|}}$.

The second property of S_γ comes from the fact that p^λ_γ is generic over the imitation of P^s_τ in b, so the result follows for a $\beta \leqslant \alpha^*$; it remains to show it for $\beta \in]\alpha^*, \mu'[$ (since $L_{\mu'+1} \vDash |p^\lambda_\gamma| = \gamma$) but this is exactly as in Lemma 1 in Section 2.2.

This achieves the proof of Theorem 1.

2.6.

(1) Following exactly the proof of Theorem 0.2 in [1] we see that the real in my theorem I may be assumed to be in L($0^\#$).

(2) To prove that the real is not provably Π^1_2 singleton, do as follows: Start from M_1 (the $\prod_n P(n)$ generic extension of M_0) and choose M'_2 to be a $C_\emptyset \times C_\emptyset$ generic extension of M_1; then there are two different reals a and a' that satisfy φ so $L(a, a') \nvDash \exists!\, x\, \varphi(x)$.

(3) The following gives a slight improvement of Theorem 1: I may assume — as in [7] — that the trees are such that: if $\alpha = |T|$ and B_0, B_1 are distinct, cofinal branches in T, then $\bar{\bar{\alpha}}^{L(B_0, B_1)} < \alpha$ (simply use an elementary, saturated — in the sense of model theory — extension of the Rational numbers); I may assume too that the first level in T only has two points: the right and the left one.

When forcing with the $P(n)$ I may assume that all the branches are left branches (i.e.: their first point is the left one) and so in φ it can be said that the branches are left ones.

Now in $L(a)$ some trees remain Suslin; add in these trees right branches. Let N be this extension. In N a remains Π^1_2 singleton since if x is another real satisfying φ, then for some i, $T(i)$ will have two branches: a right and a left one, and this is impossible.

More generally if $\bar{N}$ is an extension of N and if there is another real x in $\bar{N}$ that satisfies φ, then a class of cardinals are collapsed (in the context I prove the theorem it is $\{\chi_{n+2}(\beta) \mid \beta$ limit cardinal and — something as — $a(n) \neq x(n)\}$). I may assume that this class is $\{\alpha^+ \mid \alpha$ regular$\}$.

It would be more interesting if I could assume that all the successor cardinals were collapsed since then an extension of N where a is not Π^1_2 would have to contain $0^{\#}$; but it seems difficult to do that: it would be necessary to build trees of length successors of singular cardinals: it does not seem difficult to build such trees which are Suslin and have at most one branch (as far as $|\tau|$ is not collapsed); but in the proof I also need the fact that: (1) the trees are closed enough (all the branches of 'small' length have extension in the tree); (2) the tree remain Suslin in some extension of $L \cdots$ and I do not know how to do that!

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